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ON THE STRUCTURE OF THE SET OF SOLUTIONS OF EQUATIONS INVOLVING A-PROPER MAPPINGS(1)

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ABSTRACT. Let X and Y be Banach spaces having complete projection schemes (say, for example, they have Schauder bases). We consider various properties of mappings $T: D \subset X \to Y$ which are either Approximation-proper (A-proper) or the uniform limit of such mappings. In §1 general properties, including those of the generalized topological degree, of such mappings are discussed. In §2 we give sufficient conditions in order that the solutions of an equation involving a nonlinear mapping be a continuum. The conditions amount to requiring that the generalized topological degree not vanish, and that the mapping involved be the uniform limit of well structured mappings. We devote §3 to proving a result connecting the topological degree of an A-proper Fréchet differentiable mapping to the degree of its derivative. Finally, in §4, various Lipschitz-like conditions are discussed in an A-proper framework, and constructive fixed point and surjectivity results are obtained.

Introduction. The object of this paper is to develop further the theory of the class of A-proper mappings, a rather large class which had its origins in the constructive approximation of solutions of equations involving nonlinear operators.

In §1 we define the concepts to be used in the sequel, and review some useful results.

In §2 we prove a result which guarantees that the set of solutions of equations involving certain nonlinear operators forms a continuum. Using this result we are able to unify and extend a number of previous results in this direction; in particular, we obtain the Krasnoselsky and Sobolevsky [17] result for compact displacements, and the results of Deimling [8] and Vidossich [34] for P-compact mappings.

Our main concern in §3 is to prove a result concerning the computability of the topological degree of nonlinear A-proper mappings in terms of the degree of a linear A-proper mapping. The result of Krasnoselsky [16] concerning the computability of the Leray and Schauder degree follows as a particular case.

In §4 the general problem of knowing what conditions must be placed on a Banach space X, a set $D \subset X$, and a contraction $C: \overline{D} \subset X \to X$, in order to ensure that $I - C: \overline{D} \subset X \to X$ is A-proper, is discussed. Theorem 4.1 represents

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a partial answer, and from it we derive, in a constructive manner, a number of fixed point theorems.

1. **A-proper mappings.** In what follows X and Y will always denote real separable Banach spaces.

Definition 1.1. *A quadruple of sequences* $\Gamma = \Gamma(\{X_n\},\{Y_n\},\{P_n\},\{Q_n\})$ *is a complete projection scheme for mappings from* X *to* Y *providing that for each positive integer* n, X_n *and* Y_n *are subspaces of* X *and* Y, *respectively, of the same finite dimension,* P_n *and* Q_n *are bounded linear projections of* X *onto* X_n *and* Y *onto* Y_n, *respectively, and* $\langle P_n(x)\rangle \to x$ *and* $\langle Q_n(y)\rangle \to y$, *for each* x *in* X *and* y *in* Y.

If Γ is a complete projection scheme for mappings from X to Y and $T: D \subset X \to Y$, with $y \in Y$, then one may seek solutions of the equation

$$T(x) = y \qquad (x \in D) \tag{1.1}$$

as limits of solutions of the approximating equations

$$Q_n T(x) = Q_n(y) \qquad (x \in D \cap X_n). \tag{1.2}$$

In a series of articles W. V. Petryshyn ([22], [23], [24]) investigated the type of mappings for which the above constructive method was valid. In [25] Petryshyn introduced a wide class of mappings, those mappings satisfying condition (H), which proved to be very suitable for study by the above methods. In a later article [26] and subsequently, mappings satisfying condition (H) have been referred to as *Approximation-proper* (A-proper).

Definition 1.2. *Let* X *and* Y *be real Banach spaces with* $\Gamma = (\{X_n\},\{Y_n\},\{P_n\},\{Q_n\})$ *a complete projection scheme for mappings from* X *to* Y. *Let* $T: D \subset X \to Y$. *Then* T *is said to be A-proper with respect to* Γ *if whenever* $\langle n_k\rangle$ *is a sequence of integers and* $\langle x_{n_k}\rangle$ *is a bounded sequence with* $x_{n_k} \in D \cap X_{n_k}$, *for each* k, *and* $Q_{n_k} T(x_{n_k}) \to y$, *then there exists a subsequence* $\langle x_{n_{k(j)}}\rangle$ *of* $\langle x_{n_k}\rangle$ *which converges to* $x \in D$ *and* $T(x) = y$.

Remark 1.1. From now on we shall refer to mappings as A-proper, assuming that a complete projection scheme Γ has been given, and will remain fixed for the specific ensuing discussion.

Let us recall that if X and Y are Banach spaces and $T: D \subset X \to Y$, then T is called proper if $T^{-1}(C)$ is compact whenever C is compact. The following result of Petryshyn [27] relates this notion to that of A-properness.

Proposition 1.1. *Let* X *and* Y *be Banach spaces and let* $D \subset X$ *be open and bounded. Then if* $T: \overline{D} \subset X \to Y$ *is continuous and A-proper, it is proper.*

It is not true, as the following example shows, that a continuous proper mapping is A-proper.

Example. Let $X = l^2$, and for each positive integer n let x^n denote the element of l^2 whose nth coordinate is 1 and all of whose other coordinates are 0. Then $\{x^n\}$ forms a Schauder basis for l^2, and hence generates in a natural manner a complete projection scheme for mappings in l^2.

Now define $T: X \to X$ by $T(\langle x_n \rangle) = \langle 0, x_1, x_2, x_3, \ldots \rangle$. It is clear that T is continuous and proper; however, it is not A-proper. In fact, consider the sequence $\langle x^n \rangle$. Since $T(x^n) = x^{n+1}$ and $P_n(x^{n+1}) = 0$, we see that $P_n T(x^n) \to 0$. Since $\langle x^n \rangle$ does not have a Cauchy subsequence we see that T cannot be A-proper.

Definition 1.3. *Let X and Y be Banach spaces and let $T: D \subset X \to Y$. Then T is said to be compact if T is continuous and if $T(C)$ is precompact whenever $C \subset D$ is bounded.*

Proposition 1.2 [25]. *Let X and Y be real Banach spaces and suppose $T: D \subset X \to Y$ is A-proper. Then if $C: D \subset X \to Y$ is compact, $T + C$ is A-proper. In particular if X is a Banach space such that there is a complete projection scheme for mappings from X to X, and $D \subset X$ is closed, with $C: D \to X$ compact, then $I + C: D \subset X \to X$ is A-proper. (When $X = Y$, we let $P_n = Q_n$.)*

Remark 1.2. Let X and Y be real Banach spaces, with $\Gamma(\{X_n\}, \{Y_n\}, \{P_n\}, \{Q_n\})$ a complete projection scheme for mappings from X to Y. Suppose $T: D \subset X \to Y$. We shall use the following notation: $D_n = D \cap X_n$; and $T_n: D_n \to Y_n$ denotes the mapping $x \mapsto Q_n T(x)$, for $x \in D_n$. If $A \subset X$, $\bar{A}$ denotes its strong closure and $\dot{A}$ its boundary.

In [5], [6], Browder and Petryshyn introduced the notion of topological degree for continuous A-proper mappings defined on the closure of an open bounded set. However, it is not necessary to assume T is continuous, but instead only that T_n is continuous for sufficiently large n. In addition, one need only require that if $T: \bar{D} \subset X \to Y$, then D_n is open for all n. The Hilbert cube $C \subset l^2$ is such that C_n is open for all n, but C itself is not open. From now on we shall assume an orientation is assigned to the projection schemes being considered .

Definition 1.4. *Let X and Y be real Banach spaces and let $D \subset X$ be bounded with D_n open for each n. Let $T: \bar{D} \subset X \to Y$ be A-proper and such that $T_n: \bar{D}_n \subset X_n \to Y_n$ is continuous for each n. Let $Z' = Z \cup \{+\infty, -\infty\}$, where Z denotes the integers. Then if $g \notin T(\dot{D})$ we define* $\mathrm{Deg}(T, D, g)$ *the degree of T on D over g, to be the subset of Z'* defined as follows:

(1) *The integer m lies in* $\mathrm{Deg}(T, D, g)$ *provided there exists an infinite sequence $\langle n_k \rangle$ of positive integers such that* $\deg(T_{n_k}, D_{n_k}, Q_{n_k}(g))$ *is well defined and equals m for each k.*

(2) $+\infty$ $(-\infty)$ $\in \mathrm{Deg}(T, D, g)$ *provided there exists an infinite sequence of integers $\langle n_k \rangle$ such that* $\deg(T_{n_k}, D_{n_k}, Q_{n_k}(g))$ *is well defined for each k, and* $\lim_k \deg(T_{n_k}, D_{n_k}, Q_{n_k}(g)) = +\infty$ $(-\infty)$.

Remark 1.3. $\deg(T_n, D_n, Q_n(g))$ denotes the Brouwer degree for mappings acting between oriented Euclidean spaces of the same finite dimension.

Utilizing the properties of the Brouwer degree and of A-proper mappings, the following results were obtained in [5], [6].

Property 1.1. *If T and D are as in Definition* 1.4, *and $g \in Y$ is such that $g \notin T(\dot{D})$, then* $\mathrm{Deg}(T, D, g) \neq \varnothing$, *and if* $\mathrm{Deg}(T, D, g) \neq \{0\}$ *then there exists $x \in D$ such that $T(x) = g$.*

Property 1.2. *Let X, Y, and D be as in Definition* 1.4, *and let H*: $[0,1] \times \bar{D} \to Y$. *Suppose that for each $n \in N$, $H_n = Q_n H$: $[0,1] \times \bar{D}_n \to Y_n$ is continuous in t, uniformly with respect to $x \in \dot{D}_n$, and for each $t \in [0,1]$, $H_n(t,\cdot)$: $\bar{D}_n \to Y_n$ is continuous. Assume $H(t,\cdot)$ is A-proper for each $t \in [0,1]$, and H satisfies the following condition*:

(†) *If $\langle n_k \rangle$ is a sequence of positive integers with $\langle t_{n_k} \rangle \subset [0,1]$ and $\langle x_{n_k} \rangle \subset X$ corresponding sequences such that $x_{n_k} \in \dot{D}_{n_k}$, for each k, and $\langle Q_{n_k} H(t_{n_k}, x_{n_k}) \rangle \to f \in Y$, then $\langle n_k \rangle$ has a subsequence $\langle n_{k(j)} \rangle$ such that*

$$\langle t_{n_{k(j)}} \rangle \to t, \quad \langle x_{n_{k(j)}} \rangle \to x, \quad \text{and} \quad H(t,x) = f.$$

Then if $g \in Y$ is such that $H(t,x) \neq g$, for $t \in [0,1]$ and $x \in \dot{D}$, $\mathrm{Deg}(H(t,\cdot), D, g)$ *is independent of $t \in [0,1]$.*

Property 1.2′. *Let X, Y, and D be as in Definition* 1.4, *and let H*: $[0,1] \times \bar{D} \to Y$. *Suppose that H is continuous in t, uniformly with respect to $x \in \dot{D}$, and that for each $t \in [0,1]$, $H(t,\cdot)$ is A-proper with $H(t,\cdot)_n$: $\bar{D}_n \to Y_n$ continuous for each n. Then if $g \in Y$ is such that $H(t,x) \neq g$ for $t \in [0,1]$ and $x \in \dot{D}$,* $\mathrm{Deg}(H(t,\cdot), D, g)$ *is independent of $t \in [0,1]$.*

Remark 1.4. The invariance of degree under homotopy result stated in [5], [6] is essentially Property 1.2′ above. However, by using the properties of the Brouwer degree and of A-proper mappings one may see that Property 1.2, which contains Property 1.2′ as a special case, is true.

Property 1.3. *Let $D = D_1 \cup D_2$, where D_1 and D_2 are bounded subsets of X, with $D_i \cap X_n$ open for all $n \in N$, $i = 1, 2$. Assume T: $\bar{D} \to Y$ is A-proper, has the property that T_n: $\bar{D}_n \to Y_n$ is continuous for each n, and is such that $g \notin T(D')$, where $D' = (D_1 \cap D_2) \cup \dot{D}_1 \cup \dot{D}_2$. Then*

$$\mathrm{Deg}(T,D,g) \subset \mathrm{Deg}(T,D_1,g) + \mathrm{Deg}(T,D_2,g),$$

with equality holding if either of the right hand summands consists of a single integer. (If A and B are subsets of Z' we define $A + B = \{a \mid a = a_1 + a_2, a_1 \in A, a_2 \in B\}$, and use the convention $+\infty + (-\infty) = Z'$.)

Property 1.4. *Let $D \subset X$ be bounded, with D_n open, symmetric and containing the origin of each n. Suppose T: $\bar{D} \to Y$ is A-proper, with T_n: $\bar{D}_n \to Y_n$ continuous and $T_n|_{\dot{D}_n}$ odd for each n. Then if $0 \notin T(\dot{D})$,* $\mathrm{Deg}(T,D,0)$ *does not contain any even integers, and in particular* $\mathrm{Deg}(T,D,0) \neq \{0\}$.

The author [11] and Browder [1] independently extended the notion of topological degree to certain classes of uniform limits of A-proper mappings.

Definition 1.5. *If $D \subset X$ is open and bounded, and S: $\bar{D} \subset X \to Y$ is such that $S(\bar{D})$ is bounded, then T: $\bar{D} \subset X \to Y$ is called A-proper with respect to S providing $T + \lambda S$ is A-proper for each $\lambda > 0$ and $Q_n(T + \lambda S)$: $\bar{D} \cap X_n \to Y_n$ is continuous for each integer n.*

It was proven in [11] that if T is A-proper with respect to S and $g \in Y - \overline{T(\dot{D})}$, then for small $\beta > 0$, $\mathrm{Deg}(T + \beta S, D, g)$ is well defined and independent of β. We denote this degree by $\mathrm{Deg}_S(T,D,g)$. For this generalized notion of degree, Property 1.3 is valid, and Property 1.1 is valid providing we assume $T(\overline{D})$ is closed. Furthermore, we have the following result, whose proof will appear elsewhere.

Proposition 1.3. *Let X, Y, D, S and T be as in Definition 1.5. Suppose $g \in Y - \overline{T(\dot{D})}$. Then there exists a $\gamma > 0$ such that if L: $\overline{D} \to Y$ is A-proper with respect to S and $\|T(x) - L(x)\| \leq \gamma$, for all $x \in \dot{D}$, then* $\mathrm{Deg}_S(T, D, g) = \mathrm{Deg}_S(L, D, g)$.

By utilizing Proposition 1.3 it is easy to see that Property 1.2′ is valid for the generalized degree.

2. **On the structure of the set of solutions to nonlinear equations.** If X and Y are real Banach spaces and T: $D \subset X \to Y$, then for $y \in Y$ we may consider the equation

$$T(x) = y, \qquad (x \in D). \tag{2.1}$$

When equation (2.1) does have a solution it is of interest to know the structure of the set of solutions. Clearly the simplest situation is when the set of solutions consists of a single point; namely, there is a unique solution of equation (2.1). In this section we shall give conditions which will guarantee that the set of solutions of equation (2.1) is a continuum (i.e., it is nonempty, compact, and connected). Our main result will be Theorem 2.1, and from this theorem we shall derive as special cases various results of Krasnoselsky and Sobolevsky [17], Vidossich [34], Deimling [8], and also some new results.

Theorem 2.1. *Let X and Y be Banach spaces with $D \subset X$ open and bounded, and suppose S: $\overline{D} \to Y$ is such that $S(\overline{D})$ is bounded. Assume T: $\overline{D} \to Y$ is A-proper with respect to S, continuous, and proper. Furthermore, assume there exists a sequence $\langle T^k \rangle$ of mappings, with each T^k: $\overline{D} \to Y$ A-proper with respect to S, and satisfying*:

(1) $\mathrm{Sup}\{\|T^k(x) - T(x)\| \mid x \in \overline{D}\} = \delta_k$, *for each k, and* $\langle \delta_k \rangle \to 0$.

(2) *T^k is one-to-one, when restricted to* $(T^k)^{-1}(\overline{B}(0, \delta_k))$.

(3) *Whenever $\mathcal{O} \subset D$ is open with* $\mathrm{Deg}_S(T^k, \mathcal{O}, g) \neq \{0\}$, *then there exists $x \in \mathcal{O}$ with $T(x) = g$.*

Then if $\mathrm{Deg}_S(T, D, 0) \neq \{0\}$, *$T^{-1}(0)$ is a continuum.*

Remark 2.1. The following proof is based upon a technique of Krasnoselsky and Sobolevsky [17]. We have used superscripts to denote the approximating sequence in order to distinguish this sequence from the mappings T_n: $\overline{D}_n \to Y_n$, and T_n^k: $\overline{D}_n \to Y_n$.

Proof. Since T is proper and continuous $T(\overline{D})$ is closed, and hence, since $\mathrm{Deg}_S(T, D, 0) \neq \{0\}$, $T^{-1}(0)$ is nonempty and compact. The proof that $T^{-1}(0)$ is connected will be by contradiction. Suppose $T^{-1}(0)$ is not connected. Then we can choose disjoint closed sets N_1 and N_2, with $T^{-1}(0) = N_1 \cup N_2$, and open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ lying in D, with disjoint closures, such that $N_i \subset \mathcal{O}_i$ for $i = 1, 2$. Hence by the extension of Property 1.3 to the generalized degree, we have

$$\mathrm{Deg}_S(T, D, 0) \subset \mathrm{Deg}_S(T, \mathcal{O}_1, 0) + \mathrm{Deg}_S(T, \mathcal{O}_2, 0).$$

We shall now show that $\mathrm{Deg}_S(T, \mathcal{O}_i, 0) = \{0\}$ for $i = 1, 2$, and hence obtain a contradiction.

Now $T(x) \neq 0$ for $x \in \dot{\mathcal{O}}_1$ and since $T(\dot{\mathcal{O}}_1)$ is closed, hence $0 \notin \overline{T(\dot{\mathcal{O}}_1)}$, we may choose $\gamma > 0$ such that if $L: \overline{D} \to Y$ is A-proper with respect to S and $\|T(x) - L(x)\| \leq \gamma$, for all $x \in \dot{\mathcal{O}}_1$, then

$$\mathrm{Deg}_S(T, \mathcal{O}_1, 0) = \mathrm{Deg}_S(L, \mathcal{O}_1, 0).$$

Choose a positive integer n such that $2\delta_n < \gamma$ and let $a \in N_2$. Define $F: \overline{\mathcal{O}}_1 \to Y$ by

$$F(x) = T^n(x) - T^n(a), \quad \text{for } x \in \overline{\mathcal{O}}_1.$$

Note that $\|T^n(a)\| = \|T^n(a) - T(a)\| \leq \delta_n$, and consequently $\|T(x) - F(x)\| \leq \gamma$, for $x \in \overline{\mathcal{O}}_1$. Thus,

$$\mathrm{Deg}_S(T, \mathcal{O}_1, 0) = \mathrm{Deg}_S(F, \mathcal{O}_1, 0).$$

This implies $\mathrm{Deg}_S(T, \mathcal{O}_1, 0) = \{0\}$. Indeed, suppose this were not the case. Then $\mathrm{Deg}_S(F, \mathcal{O}_1, 0) \neq \{0\}$, and hence we may choose $\tilde{x} \in \mathcal{O}_1$, with $F(\tilde{x}) = 0$, i.e. $T^n(\tilde{x}) = T^n(a)$. Since $\tilde{x} \neq a$ and $\|T^n(a)\| \leq \delta_n$, we have a contradiction. Consequently we have $\mathrm{Deg}_S(T, \mathcal{O}_1, 0) = \{0\}$, and in similar fashion one shows $\mathrm{Deg}_S(T, \mathcal{O}_2, 0) = \{0\}$.

Hence, under the assumption that $T^{-1}(0)$ is not connected we have shown that $\mathrm{Deg}_S(T, D, 0) = \{0\}$. It follows that $T^{-1}(0)$ is connected, and since it is compact, is a continuum. Q.E.D.

When a mapping is A-proper and the Browder-Petryshyn degree is defined it coincides with the generalized degree. Hence one obtains the following result from Theorem 2.1.

Proposition 2.1. *Let X, Y, and D be as in Theorem* 2.1. *Suppose $T: \overline{D} \to Y$ is continuous and A-proper. Assume there exists a sequence $\langle T^k \rangle$ of mappings, with each $T^k: \overline{D} \to Y$ A-proper and T_n^k continuous, and satisfying* (1) *and* (2) *of Theorem* 2.1. *Then if* $\mathrm{Deg}(T, D, 0) \neq \{0\}$, *$T^{-1}(0)$ is a continuum.*

Remark 2.2. The above Proposition 2.1 is true if we weaken the assumption that T is continuous to the requirement that T is proper and T_n is continuous for each n. This follows from the observation that if $T(x) \neq 0$ for $x \in \dot{\mathcal{O}}$, where $\mathcal{O} \subset D$ is open, then $\mathrm{Deg}(T, \mathcal{O}, 0)$ is well defined. We add that Deimling [8] has proven a theorem similar to Proposition 2.1, under the additional assumption that $X = Y$.

In proving our first Corollary of Theorem 2.1 we will need the following known result.

Proposition 2.2. *Let X be a Banach space and suppose C: $\overline{B}(0,1) \subset X \to Y$ is compact. Let $T = I - C$: $\overline{B} \to X$ be such that T is one-to-one and $T(0) = 0$. Then the Leray-Schauder degree of T on B over 0 is nonzero (see [7]). Furthermore, in this case the generalized degree is a set consisting of a single integer equal to the Leray-Schauder degree (see [5]), and consequently* $\operatorname{Deg}(T, B, 0) \neq \{0\}$.

Proposition 2.3. *Let X be a Banach space, with $x_0 \in X$ such that C: $\overline{B}(x_0, r) \subset X \to X$ is compact. Assume $T = I - C$: $\overline{B} \subset X \to X$ is such that $y_0 \notin T(\dot{B})$ where $y_0 = T(x_0)$. Suppose there exists a sequence $\langle T^k \rangle$, with $T^k = I - C^k$: $\overline{B} \subset X \to X$ for each k, such that each C^k is compact and each T^k is a homeomorphism. Then if $\langle T^k \rangle$ converges uniformly to T, $T^{-1}(y_0)$ is a continuum.*

Proof. We shall assume that $x_0 = 0$ and $T(x_0) = 0$. When we prove the result in this special case the general result follows by considering $\tilde{T}$: $\overline{B}(0,r) \to X$ defined by $\tilde{T}(x) = T(x_0 - x) - T(x_0)$ and $\tilde{T}^k$: $\overline{B}(0,r) \to X$ defined by $\tilde{T}^k(x) = T^k(x_0 - x) - T^k(x_0)$, and noting that $\{x_0 - x \mid x \in \overline{B}(0,r), \tilde{T}(x) = 0\} = \{x \mid x \in \overline{B}(x_0, r), T(x) = y_0\}$.

Hence, under the assumption that $T(x) \neq 0$, for all $x \in \dot{B}(0,r)$ we may, by Proposition 1.3, choose $c > 0$ such that if L: $\overline{B} \to X$ is A-proper and each L_n is continuous and $\|L(x) - T(x)\| \leq c$ for all $x \in \overline{D}$, then $\operatorname{Deg}(L, D, 0) = \operatorname{Deg}(T, D, 0)$. Now choose $n_0 \in N$ such that $\operatorname{Sup}\{\|T(x) - T^{n_0}(x)\| \mid x \in \overline{B}\} < c/2$. Then since $T(0) = 0$ we must have $\|T^{n_0}(0)\| < c/2$, and consequently if we define L: $B \subset X \to X$ by $L(x) = T^{n_0}(x) - T^{n_0}(0)$ then L is one-to-one, $L(0) = 0$, and since $L = I - C^{n_0} - C^{n_0}(0)$ and $C^{n_0} - C^{n_0}(0)$ is compact, we may invoke Proposition 2.2 to show $\operatorname{Deg}(L, B, 0) = \{0\}$, and consequently $\operatorname{Deg}(T, B, 0) \neq \{0\}$. Since all of the hypotheses of Theorem 2.1 are satisfied we know $T^{-1}(0)$ is a continuum. Q.E.D.

The next result, which was in fact the motivation for Theorem 2.1, was first proven in the work of Krasnoselsky and Sobolevsky [17]. It is an immediate consequence of Theorem 2.1.

Proposition 2.4. *Let X be a Banach space and let $D \subset X$ be bounded and open. Suppose C: $D \subset X \to X$ is compact and such that* $\operatorname{Deg}(I - C, D, 0)$ *is well defined and nonzero. Suppose there exists $\langle C^k \rangle$, with C^k: $\overline{D} \to X$ compact and such that $\|C^k(x) - C(x)\| \leq \delta_k$ for all $x \in \overline{D}$ and for all $k \in K$, with $\delta_k \to 0$ as $k \to \infty$. Furthermore, suppose that for each $y \in Y$ and $k \in N$ with $\|y\| \leq \delta_k$, the equation*

$$x - C^k(x) = y, \qquad (x \in \overline{D})$$

has at most one solution. Then $(I - C)^{-1}(0)$ is a continuum.

In order to prove our next result we shall introduce the class of duality mappings.

Definition 2.1. *A gauge function is a real-valued continuous function* μ *defined on the nonnegative half-line* R^+ *such that* (a) $\mu(0) = 0$, (b) $\lim_{t\to\infty} \mu(t) = +\infty$, (c) μ *is strictly increasing. Suppose* X *is a Banach space and* X^* *is the space of bounded linear functionals on* X. *Then the duality mapping* J *with gauge function* μ *from* X *to* 2^{X^*} *is defined by*:

$$J(x) = \{f \in X^* \mid \|f\| = \mu(\|x\|), (f,x) = \mu(\|x\|) \cdot \|x\|\},$$

where the notation (f,x) *means the functional* f *evaluated at the point* $x \in X$.

Definition 2.2. *Let* X *be a Banach space and suppose* $T\colon A \subset X \to X$. *If* J *is a duality mapping defined on* X, *then* T *is said to be accretive (sometimes called J-monotone) provided* $(T(x) - T(y), f) \geq 0$, *for all* x, y *in* A *and* $f \in J(x-y)$.

Accretive mappings, which were first introduced by Browder [3], are a generalization to arbitrary Banach spaces of monotone mappings defined on a Hilbert space.

Definition 2.3. *Let* X *be a Banach space with* $\Gamma = (\{X_n\},\{P_n\})$ *a complete projection scheme for mappings from* X *to* X. *Let* $T\colon G \subset X \to X$ *and let* $\nu \in R$ *be nonnegative. Then* T *is said to be* P_ν*-compact provided that for every* α *dominating* ν *(i.e.,* $\alpha \geq \nu$, *if* $\nu > 0$ *and* $\alpha > \nu$, *if* $\nu = 0$*) the mapping* $T - \alpha I\colon G \subset X \to X$ *is A-proper.*

The class of P_0-compact mappings, usually called simple P-compact mappings, was introduced and studied by Petryshyn (see [28]), and in fact was the precursor of the class of A-proper mappings. In [29] Petryshyn introduced the class of P_ν-compact mappings for $\nu > 0$, and their study was continued by Petryshyn and Tucker [31].

We are now in a position to state our next result.

Proposition 2.5. *Let* X *be a Banach space and suppose* $D \subset X$ *is open and bounded. Suppose* $T\colon \overline{D} \subset X \to X$ *is* P_1*-compact with* T_n *continuous for each* n, *and such that* $I - T\colon \overline{D} \subset X \to X$ *is J-monotone with respect to some duality mapping* J *and proper. Then if* $\operatorname{Deg}(I - T, D, 0)$ *is well defined and* $\neq \{0\}$, $F(T) = \{x \mid T(x) = x\}$ *is a continuum.*

Proof. Since T is P_1-compact we see that $T - \lambda I$ is A-proper for each $\lambda \geq 1$. Consequently, $\lambda I - T$ is A-proper for each $\lambda \geq 1$. Now define $T^k\colon \overline{D} \to X$ by $T^k(x) = (1 + 1/k)(x) - T(x)$. Thus, for each $k \in N$, T^k is A-proper, and such that T_n^k is continuous. Furthermore, we see that $(T^k(x) - T^k(y), f) \geq (1/k)\mu(\|x - y\|) \cdot \|x - y\| > 0$ for $x, y \in \overline{D}, f \in J(x-y)$, where $x \neq y$. Hence T^k is one-to-one for each k. Since T^k converges uniformly to $I - T$ on $\overline{D}$ we may invoke Theorem 2.1 to conclude that $F(T)$ is a continuum. Q.E.D.

Corollary 2.1. *Let* X *be a Banach space and suppose* $D \subset X$ *is open and bounded. Suppose* $T\colon \overline{D} \to X$ *is* P_1*-compact, with* $T(\dot{D})$ *bounded and each* T_n *continuous.*

Assume also that $I - T$ is proper and J-monotone. Then if $F(T) \neq \varnothing$ and $F(T) \cap \dot{B} = \varnothing$, $F(T)$ is a continuum.

Proof. We may assume, without loss of generality, that $0 \in D$ and $T(0) = 0$. Now our conclusion will follow immediately from the previous result if we can show $\mathrm{Deg}(I - T, D, 0) \neq \{0\}$.

Let $H: [0,1] \times \overline{D} \to X$ be defined by $H(t,x) = t(x - T(x)) + (1 - t)x$, for $x \in \overline{D}$ and $t \in [0,1]$. It is clear that for each $t \in [0,1]$, $H(t, \cdot)$ is A-proper, with $H(t, \cdot)_n$ continuous for $n \in N$. Furthermore, since $T(\dot{D})$ is bounded, H is continuous in t, uniformly with respect to $x \in \dot{D}$.

Now for $t \in [0,1)$ and $x \in \dot{D}$,

$$(H(t,x), J(x)) \geq (1 - t)(x, J(x)) > 0.$$

Thus, since we have assumed $H(1,x) \neq 0$ for $x \in \dot{D}$, it follows that $H(t,x) \neq 0$ for $x \in \dot{D}$ and $t \in [0,1]$. Hence, by Property 1.2′ we see that $\mathrm{Deg}(I - T, D, 0) = \mathrm{Deg}(I, D, 0) \neq \{0\}$. Q.E.D.

The most obvious examples of P_1-compact mappings are compact mappings. It is also clear that if C is compact then $I - C$, when defined on a closed set, is proper. With this in mind we immediately obtain the next result.

Corollary 2.2. *Let X be a Banach space, and suppose $D \subset X$ is open and bounded. Let $T: \overline{D} \to X$ be compact, with $I - T$ J-monotone. Then if $F(T) \neq \varnothing$ and $F(T) \cap \dot{D} = \varnothing$, $F(T)$ is a continuum.*

In [8] Deimling proved a slightly less general result then our next proposition. As our proof shows, it is a direct consequence of Theorem 2.1.

Proposition 2.6. *Let $D \subset X$ be open and bounded, and suppose $T: \overline{D} \subset X \to X$ is continuous, P_1-compact, and such that $T(\dot{D})$ is bounded. Assume there is some x_0 in D such that*

(†) if $T(x) - x_0 = \lambda(x - x_0)$ with $x \in \dot{D}$, then $\lambda < 1$.(2) Assume also that there is a sequence $\langle T^k \rangle$ of A-proper mappings, each $T^k: \overline{D} \to X$ continuous, with

$$\mathrm{Sup}\{\|T^k(x) - T(x)\| \mid x \in \overline{D}\} = \delta_k \quad \textit{where } \delta_k \to 0 \textit{ as } k \to \infty,$$

and T^k one-to-one on $(T^k)^{-1}(B(0, \delta_k))$. Then $\{x \mid T(x) = x, x \in D\}$ is a continuum.

Proof. It is clear that it suffices to show $\mathrm{Deg}(I - T, D, 0) \neq \{0\}$, for then we may invoke Theorem 2.1.

Consider $H: [0,1] \times \overline{D} \to X$, defined by $H(t,x) = (x - x_0) - t(T(x) - x_0)$ for $t \in [0,1]$, $x \in \overline{D}$. Then $H(t, \cdot)$ is A-proper for each $t \in [0,1]$, and H is continuous in t, uniformly with respect to $x \in \dot{D}$. Now, by condition (†),

(2) We denote this condition by $\Pi_1^{<}(x_0)$, with $\Pi_1^{<}$ meaning $\Pi_1^{<}(0)$.

$H(t,x) \neq 0$ for $x \in \dot{D}$. Thus, using Property 1.2′, we have $\mathrm{Deg}(I - T, D, 0) = \mathrm{Deg}(I - x_0, D, 0)$, and since the latter degree is $\neq \{0\}$, our result is proven. Q.E.D.

Definition 2.4. *Let X be a Banach space and suppose $T\colon A \subset X \to X$. Then T is said to be nonexpansive provided*

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \textit{for all } x, y \textit{ in } A.$$

Now it is clear that if $T\colon A \subset X \to X$ is nonexpansive and if J is any duality mapping defined on X then $I - T$ is J-monotone. Thus the results we have derived here are applicable to the study of fixed points of nonexpansive mappings. However, this is of interest only in the case when X fails to be strictly convex, for when X is strictly convex and D is convex, closed, and bounded then Schaefer [32] has shown the fixed points of a nonexpansive mapping form not only a connected set, but a convex set. (Recall that a Banach space is strictly convex if the boundary of the unit ball does not contain any line segments.) As DeMarr has shown, when X is not strictly convex this is not always the case.

Proposition 2.7. *Let X be a Banach space with $D \subset X$ a bounded, open set which contains the origin. Suppose $T\colon \overline{D} \subset X \to X$ is nonexpansive, P_1-compact, and satisfies $\Pi_1^{<}$. Then $F(T)$ is a continuum.*

Proof. The proof follows immediately from our remarks on nonexpansive mappings, P_1-compact mappings, and Proposition 2.6. Q.E.D.

By use of Corollary 2.2 and our remarks on nonexpansive mappings one obtains the following corollary, which first appeared as Theorem 2.2 in Vidossich [34], under the assumption $D = B(x_0, r)$ and $C(x_0) = x_0$.

Corollary 2.3. *Let X be a Banach space with $D \subset X$ open and bounded. Let $C\colon \overline{D} \to X$ be compact and nonexpansive. Then if $F(C) \neq 0$, and $F(C) \cap \dot{B} = \varnothing$, $F(C)$ is a continuum.*

Remark 2.3. Since T is P_1-compact with each T_n continuous iff $I - T$ is A-proper with respect to I and also A-proper, a number of the previous results may be proven, with suitable modifications, without the assumption that $I - T$ is A-proper, where we make use of $\deg_I(I - T, D, 0)$ instead of $\deg(I - T, D, 0)$.

Remark 2.4. In [30] Petryshyn discusses the structure of fixed point sets of k-set contractions. For an extensive discussion of the properties of this class of mapping one may consult Nussbaum [19].

3. **On A-proper mappings having A-proper derivatives.** In this section we shall examine conditions which guarantee that the solutions of a nonlinear equation are isolated. In order to do so we first discuss the Fréchet differentiability of A-proper mappings.

Definition 3.1. *Let X and Y be Banach spaces and suppose $D \subset X$ is open. Let $T\colon D \subset X \to Y$. If $x_0 \in D$, then T is said to be Fréchet differentiable at x_0*

providing there exists a linear bounded operator $T'_{x_0}: X \to Y$, such that for each $h \in X$ and $t \in R$ with $x_0 + th \in D$,

$$T(x_0 + th) - T(x_0) = t \cdot T'_{x_0}(h) + w(x_0; th),$$

where $w(x_0; y)$ is such that for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|w(x_0, y)\| \leq \varepsilon \cdot \|y\|$ when $\|y\| < \delta$. The operator T'_{x_0} is called the Fréchet derivative of T at x_0.

The following result has been known for some time (see [33]).

Proposition 3.1. *Let X and Y be Banach spaces and suppose $C: X \to Y$ is compact. Then if C is Fréchet differentiable at $x_0 \in X$, C'_{x_0} is also compact.*

It was shown by Yamamuro [37] that even if one knows C'_x exists and is compact for all $x \in X$ it is not necessarily the case that C is compact. The nearest to a converse of Proposition 3.1 is the following result (see [33, p. 51]).

Proposition 3.2. *Let X and Y be Banach spaces and suppose $C: X \to Y$ is Fréchet differentiable at each $x \in X$ and furthermore C'_x is compact, for all $x \in X$. Then if the mapping $C': X \to L(X, Y)$ defined by $x \mapsto C'_x$, is compact, C is compact.*

The example of Yamamuro [37] may also be used to show that a mapping can have an A-proper Fréchet derivative at each point without being A-proper itself.

Example. Let $X = l^2$ and let $\langle x^n \rangle$ be the natural Schauder basis. Now define $C: l^2 \to l^2$ by $C(x) = \sum_{n=1}^{\infty} (x, x^n)^2 \cdot x^n$, and let $T = I - C$. We first of all note that T is not A-proper with respect to the projection scheme induced by $\langle x^n \rangle$. Indeed, since $T(x^n) = 0$ for each n, $\langle P_n T(x^n) \rangle \to 0$, and $\langle x^n \rangle$ clearly does not have any convergent subsequence. Now it is easy to verify that C has a Fréchet derivative at each $x \in X$, and

$$C'_x(y) = \sum_{n=1}^{\infty} (x, x^n)(y, x^n) x^n$$

for each $y \in X$, and consequently C'_x is compact for each x. Since $T'_x = I - C'_x$, we see T'_x is A-proper for each $x \in X$.

In order to obtain a result analogous to Proposition 3.2 for A-proper mappings one has to strengthen the continuity condition on the derivative.

Definition 3.2. *Let X and Y be Banach spaces. If $\langle x_n \rangle \subset X$ converges in the weak topology on X to x we shall use the notation $x_n \rightharpoonup x$. A mapping $T: D \subset X \to Y$ is said to be strongly continuous provided $T(x_n) \to T(x)$ whenever $\langle x_n \rangle \subset D$ is such that $x_n \rightharpoonup x \in D$.*

In [31] Petryshyn and Tucker proved an analogue of Proposition 3.2 for P-compact mappings of X into X under the assumption that the derivative is strongly continuous. An examination of their proof shows that the P-compactness is not essential, and that their technique of proof can be utilized to prove the following theorem. The continuity condition imposed is weaker than strong continuity.

Theorem 3.1. *Let X and Y be Banach spaces, with X reflexive. Suppose that T: $X \to Y$ is Fréchet differentiable and that its derivative satisfies the following continuity condition*:

(†) *If* $\langle y_n \rangle \subset X$ *and* $\langle z_n \rangle \subset X$ *with* $y_n \rightharpoonup x$ *and* $z_n \rightharpoonup x$, *then* $T'_{y_n}(x - z_n) - T'_x(x - z_n) \to 0$.

Then the following are equivalent:

(1) *T is A-proper.*

(2) *T'_x is A-proper for each $x \in X$.*

If X is a Banach space and C: $X \to X$ is compact, consider $T = I - C$. When $T(x_0) = 0$ and T is Fréchet differentiable at x_0, with T'_{x_0} one-to-one, then there exists an $r > 0$ such that $\deg(T, B(x_0, r), T(x_0)) = \deg(T'_{x_0}, B(0, r), 0)$ (see [16]). We will now generalize this result to A-proper mappings and at the same time extend results of a similar nature proven by Wong [36].

Definition 3.3. *Let X and Y be Banach spaces. Suppose T: $D \subset Y \to Y$. Let $y \in Y$ and consider the equation $T(x) = y$ ($x \in D$). Then $x_0 \in D$ is said to be an isolated solution of the above equation provided $T(x_0) = y$ and there exists an $r > 0$ such that $T(x) \neq y$ for all x with $0 < \|x - x_0\| \leq r$.*

The following lemma, whose proof follows directly from known properties of the Brouwer degree, will be needed.

Lemma 3.1. *Let X and Y be Banach spaces with $D \subset X$ open and bounded. Let T: $\overline{D} \subset X \to Y$ be A-proper, with each T_n continuous, and let $y \in Y$ and $n_0 \in N$ be such that $\|T_n(x) - Q_n(y)\| \geq c > 0$, for all $n \geq n_0$ and for all $x \in \dot{D}_n$. If S: $\overline{D} \to Y$ is A-proper, with each S_n continuous, and $\|S_n(x) - T_n(x)\| < c$, for all $x \in \dot{D}_n$ and all $n \geq n_0$, then* $\operatorname{Deg}(T, D, y) = \operatorname{Deg}(S, D, y)$, *providing both degrees are defined.*

Theorem 3.2. *Let X and Y be Banach spaces and let $D \subset X$ be open and bounded. Suppose T: $\overline{D} \subset X \to Y$ is A-proper and that at $x_0 \in D$, T is Fréchet differentiable, with T'_{x_0} one-to-one and A-proper. Then x_0 is an isolated solution of the equation $T(x) = T(x_0)$ ($x \in D$), and there exists an $r > 0$ such that*

$$\operatorname{Deg}(T, B(x_0, r), T(x_0)) = \operatorname{Deg}(T'_{x_0}, B(x_0, r), T'_{x_0}(x_0)).$$

Proof. Since T'_{x_0}: $X \to Y$ is linear, bounded, one-to-one and A-proper, there exists $c > 0$ and $n_1 \in N$ such that $\|Q_n T'_{x_0}(x)\| \geq c\|x\|$ for all $x \in X_n$, $n \geq n_1$. We know that for $x \in X$ and small t

$$T(x_0 + tx) - T(x_0) = t \cdot T'_{x_0}(x) + w(x_0; tx),$$

where $\|w(x_0; y)\|/\|y\| \to 0$ as $\|y\| \to 0$.

Now, let $M = \operatorname{Sup}\{\|Q_n\| \mid n \in N\}$,(3) and choose $r > 0$ such that $M \cdot \|w(x_0; y)\|/\|y\| < c/8$, if $\|y\| \leq r$, and such that $\overline{B}(x_0, r) \subset D$.

(3) We know $M < \infty$ because of the uniform boundedness principle.

To prove that $\mathrm{Deg}(T, B(x_0, r), T(x_0)) = \mathrm{Deg}(T'_{x_0}, B(x_0, r), T'_{x_0}(x_0))$, it suffices to show that

$$\mathrm{Deg}(T, B(x_0, r), T(x_0)) = \mathrm{Deg}(T'_{x_0} - T'_{x_0}(x_0) + T(x_0), B(x_0, r), T(x_0)).$$

Now, if $x \in \dot{B}(x_0, r) \cap X_n$, where $n \geq n_1$, we have

$$\begin{aligned}
Q_n T(x) - Q_n T(x_0) &= Q_n\{T'_{x_0}(x - x_0) + w(x_0; x - x_0)\} \\
&= Q_n\{T'_{x_0}(x - P_n(x_0)) + T'_{x_0}(P_n(x_0) - x_0) + w(x_0; x - x_0)\} \\
(\dagger\dagger) \qquad &\geq c \cdot \|x - P_n(x_0)\| - M \cdot \|T'_{x_0}\| \cdot \|P_n(x_0) - x_0\| - M \cdot \|w(x_0; x - x_0)\| \\
&= \|x - x_0\| \left\{ c \cdot \frac{\|x - P_n(x_0)\|}{\|x - x_0\|} - M \cdot \|T'_{x_0}\| \right. \\
&\qquad \left. \cdot \frac{\|P_n(x_0) - x_0\|}{\|x - x_0\|} - M \cdot \frac{\|w(x_0; x - x_0)\|}{\|x - x_0\|} \right\}.
\end{aligned}$$

Choose $n_2 \in N$ such that for each $n \geq n_2$ we have

$$\|P_n(x_0) - x_0\| \leq \min\{r \cdot c/8M \cdot \|T'_{x_0}\|, r/2\}.$$

Hence, if $n \geq n_2$ and $x \in \dot{B}(x_0, r) \cap X_n$, we have

$$\begin{aligned}
c \cdot \frac{\|x - P_n(x_0)\|}{\|x - x_0\|} &\geq c \cdot \left\{ \frac{\|x - x_0\| - \|x_0 - P_n(x_0)\|}{\|x - x_0\|} \right\} \\
&\geq c \cdot \left\{ \frac{r - r/2}{r} \right\} = \frac{c}{2}.
\end{aligned}$$

Hence from $(\dagger\dagger)$ we may conclude that $\|Q_n T(x) - Q_n T(x_0)\| \geq (r \cdot c)/4$ for all $x \in \dot{B}(x_0, r) \cap X_n$, where $n \geq \max\{n_1, n_2\}$.

We shall now estimate $\|Q_n T(x) - Q_n T'_{x_0}(x) + Q_n T'_{x_0}(x_0) - Q_n T(x_0)\|$ for $x \in \dot{B}(x_0, r) \cap X_n$, where $n \geq \max\{n_1, n_2\}$.

Well, $Q_n T(x) = Q_n T(x_0 + (x - x_0))$ and therefore

$$\begin{aligned}
&Q_n T(x) - Q_n T'_{x_0}(x) + Q_n T'_{x_0}(x_0) - Q_n T(x_0) \\
&\qquad = Q_n\{T(x_0 + (x - x_0)) - T(x_0) - T'_{x_0}(x - x_0)\} = Q_n w(x_0; x - x_0).
\end{aligned}$$

Consequently,

$$\|Q_n T(x) - \{Q_n T'_{x_0}(x) - Q_n T'_{x_0}(x_0) + Q_n T(x_0)\}\| = \|Q_n w(x_0; x - x_0)\|$$
$$\leq M \cdot \|x - x_0\| \cdot \|w(x_0; x - x_0)\|/\|x - x_0\| < r \cdot c/8,$$

for all $x \in \dot{B}(x_0, r) \cap X_n$, and $n \geq \max\{n_1, n_2\}$. Hence we see by Lemma 3.1 that

$$\mathrm{Deg}(T, B(x_0, r), T(x_0)) = \mathrm{Deg}(T'_{x_0} - T'_{x_0}(x_0) + T(x_0), B(x_0, r), T(x_0))$$

and since it is clear from the definition of the A-proper degree and from the corresponding property of the Brouwer degree that

$$\mathrm{Deg}(T'_{x_0} - T'_{x_0}(x_0) + T(x_0), B(x_0, r), T(x_0)) = \mathrm{Deg}(T'_{x_0}, B(x_0, r), T'_{x_0}(x_0)),$$

our theorem is proven. Q.E.D.

There is no analogue in the A-proper degree theory of the product theorem for the degree for compact displacements. The next result may be regarded as a product theorem for two particular types of A-proper mappings, namely, the product of a linear A-proper mapping and a translation.

Theorem 3.3. *Let X and Y be Banach spaces, and suppose $T: X \to Y$ is linear, bounded, one-to-one, and A-proper. Then if $x_0 \in X$ and $r > 0$,* $\mathrm{Deg}(T, B(0, r), 0) = \mathrm{Deg}(T, B(x_0, r), T(x_0))$.

Proof. Choose $x_0 \in X$ and $r > 0$. Since $T(x) \neq T(x_0)$, for $x \in \dot{B}(x_0, r)$ we may choose $c > 0$ and $n_1 \in N$ such that $\|Q_n T(x) - Q_n T(x_0)\| \geq c$ for $x \in \dot{B}(x_0, r) \cap X_n$ and $n \geq n_1$.

Choose $n_2 \in N$ such that $\|Q_n T(x_0) - Q_n T(P_n(x_0))\| < c$ for $n \geq n_2$.

Choose $n_3 \in N$ such that $\|P_n(x_0) - x_0\| < r$ for $n \geq n_3$.

Let $n_0 = \max\{n_i \mid i = 1, 2, 3\}$, and for the remainder of the proof let $n \in N$ with $n \geq n_0$ be fixed.

By our choice of n_1 and n_2 and by use of the homotopy theorem for the Brouwer degree we see that

$$(\dagger\dagger\dagger) \qquad \begin{aligned} &\deg(T_n - Q_n T(x_0), B(x_0, r) \cap X_n, 0) \\ &\qquad = \deg(T_n - T_n(P_n(x_0)), B(x_0, r) \cap X_n, 0). \end{aligned}$$

We also know that

$$\begin{aligned} &\deg(T_n - T_n(P_n(x_0)), B(x_0, r) \cap X_n, 0) \\ &\qquad = \deg(T_n, B(x_0, r) \cap X_n, T_n(P_n(x_0))). \end{aligned}$$

Now, let $R_n: \overline{B}(0,r) \cap X_n \to X_n$ be defined by $R_n(x) = x + P_n(x_0)$.

Then R_n is a continuous one-to-one mapping and $P_n(x_0) = R_n(0)$. Hence we see that since R preserves orientation,

$$\deg(R_n, B(0,r) \cap X_n, P_n(x_0)) = 1.$$

By our choice of n_3, $P_n(x_0) \in B(x_0,r) \cap X_n$. Consequently, by the product theorem for the Brouwer degree,

$$\deg(T_n \circ R_n, B(0,r) \cap X_n, T_n(P_n(x_0)))$$
$$= \deg(R_n, B(0,r) \cap X_n, T_n(P_n(x_0)))\deg(T_n, B(x_0,r) \cap X_n, T_n(P_n(x_0))).$$

Since T is linear, $(T_n \circ R_n)(x) = T_n(x) + T_n(P_n(x_0))$, and hence

$$\deg(T_n + T_n(P_n(x_0)), B(0,r) \cap X_n, T_n(P_n(x_0))) = \deg(T_n, B(x_0,r) \cap X_n, T_n(P_n(x_0))).$$

But

$$\deg(T_n + T_n(P_n(x_0)), B(0,r) \cap X_n, T_n(P_n(x_0))) = \deg(T_n, B(0,r) \cap X_n, 0),$$

and hence

$$\deg(T_n, B(0,r) \cap X_n, 0) = \deg(T_n, B(x_0,r) \cap X_n, T_n(P_n(x_0))).$$

Using this last equality, together with (†††) we obtain

$$\deg(T_n, B(0,r) \cap X_n, 0) = \deg(T_n - Q_n T(x_0), B(x_0,r) \cap X_n, 0)$$

and since this is true for all $n \geq n_0$, we have

$$\mathrm{Deg}(T, B(0,r), 0) = \mathrm{Deg}(T - T(x_0), B(x_0,r), 0)$$
$$= \mathrm{Deg}(T, B(x_0,r), T(x_0)). \quad \text{Q.E.D.}$$

We may now combine Theorem 3.2 and Theorem 3.3 to obtain the following generalization of a well-known theorem for compact displacements.

Corollary 3.1. *Let X and Y be Banach spaces and let $D \subset X$. Suppose that $T: \overline{D} \subset X \to Y$ is A-proper and that at $x_0 \in D$, T is Fréchet differentiable, with T'_{x_0} one-to-one and A-proper. Then there exists an $r > 0$ such that*

$$\mathrm{Deg}(T, B(x_0,r), T(x_0)) = \mathrm{Deg}(T'_{x_0}, B(0,r), 0).$$

Now it is clear that if $T: D \subset X \to Y$ is a continuous proper mapping and if $y \in Y$ is such that the solutions of $T(x) = y$ $(x \in D)$ are isolated, then the set of solutions is finite. Since continuous, A-proper mappings when restricted to bounded closed sets are proper we may combine our last observation with Theorem 3.2 to obtain the following.

Corollary 3.2. *Let X and Y be Banach spaces, with $D \subset X$ open and bounded. Suppose $T\colon \overline{D} \subset X \to Y$ is continuous and A-proper, and suppose furthermore that $y \in Y$ is such that if $x_0 \in \overline{D}$ is such that $T(x_0) = y$, then T'_{x_0} exists, is A-proper, and one-to-one. Then there are at most a finite number of solutions of*

$$T(x) = y \qquad (x \in D).$$

4. Lipschitzian and compact mappings. In recent years a good deal of attention has been paid to the proving of surjectivity theorems and to the existence of fixed points for mappings which are either Lipschitzian, compact, or the intertwining of such mappings. We shall now examine these classes of mappings from the viewpoint of A-properness, and in this manner extend some results of Kirk [15], Browder [5], and Webb [35], and also generalize in a constructive manner the classical result that a mapping of the form $I - S$, where S is a contraction defined on a Banach space, is surjective. Before proceeding, we shall fix our terminology. The reader should be aware that the terminology used by different authors varies considerably.

Definition 4.1. *Let X and Y be Banach spaces and let $S\colon D \subset X \to Y$.*

(a) *Suppose there exists $\alpha > 0$ such that*

$$\|S(x) - S(y)\| \leq \alpha \cdot \|x - y\|, \quad \textit{for all } x, y \in D.$$

Then S is said to be nonexpansive if $\alpha = 1$, and is said to be contractive if $0 < \alpha < 1$.

(b) *If*

$$\|S(x) - S(y)\| < \|x - y\|, \quad \textit{for all } x, y \in D, x \neq y,$$

then S is said to be strictly nonexpansive.

(c) *If for each $x \in D$ there exists $\alpha(x) \in R$ with $0 < \alpha(x) < 1$ such that*

$$\|S(x) - S(y)\| \leq \alpha(x) \cdot \|x - y\|, \quad \textit{for all } y \in D,$$

then S is said to be a generalized contraction.

The class of contractive mappings has been studied for many decades, and it is known that if X is a Banach space and $S\colon X \to X$ is a contraction then $I - S$ is onto.

It is not true in general that if $S\colon X \to S$ is a strictly nonexpansive mapping then $I - S$ is surjective. The following example, in fact, shows it not even true in l^2.

Example. Define $T\colon l^2 \to l^2$ as follows: For each $x \in l^2$ with $x = \langle x_n \rangle$, let

$$T(x) = \langle 1, (1/2)^{1/2} x_1, (2/3)^{1/2} x_2, \ldots, (n/(n+1))^{1/2} x_n, \ldots \rangle.$$

If $x = \langle x_k \rangle \in l^2$ and $y = \langle y_k \rangle \in l^2$, with $x \neq y$, then

$$T(x) - T(y) = \langle 0, (1/2)^{1/2}(x_1 - x_2), \ldots, (n/(n+1))^{1/2}(x_n - y_n), \ldots \rangle.$$

Choose n_0 to be an integer such that $x_{n_0} \neq y_{n_0}$. Then clearly we have

$$\|T(x) - T(y)\|^2 \leq (-1/(n_0 + 1))|x_{n_0} - y_{n_0}|^2 + \|x - y\|^2,$$

and hence

$$\|T(x) - T(y)\| < \|x - y\|.$$

Hence T is strictly nonexpansive. However $I - T$ is not onto, for one may check that the only possible $\tilde{x}$ such that $\tilde{x} - T(\tilde{x}) = 0$ is

$$\tilde{x} = \langle 1, (1/2)^{1/2}, (1/3)^{1/2}, \ldots, (1/n)^{1/2}, \ldots \rangle,$$

and clearly this $\tilde{x}$ is not in l^2.

Another well-known result for contractive mappings is that when F is a closed subset of a Banach space X and S: $F \to F$ is a contraction then S has a fixed point.

As Kirk (see [9]) has shown, this result does not generalize to strictly nonexpansive mappings even if one assumes F to be closed, convex, and bounded.

However, if one imposes additional geometric structure on X, namely assumes that X is uniformly convex, then Browder [2], Göhde [12], and Kirk [13] proved that when $F \subset X$ is closed, bounded, and convex, and S: $F \to F$ is nonexpansive, then S has a fixed point in F.

The above result is no longer valid, even in a Hilbert space with $F = \bar{B}(0, 1)$, for a mapping of type $S + C$, where S is nonexpansive and C is compact (see Browder [3]). We recall that a Banach space X is said to be uniformly convex provided that for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \varepsilon$, then $\frac{1}{2}\|x + y\| \leq 1 - \delta(\varepsilon)$. The spaces $L^p[0, 1]$, l^p, $1 < p < \infty$, and Hilbert spaces are uniformly convex, and all uniformly convex spaces are reflexive. In the past few years the Browder-Göhde-Kirk theorem has been generalized to various classes of intertwining mappings, and recently Nussbaum [20] has generalized the result to the class of locally almost nonexpansive mappings.

The notion of a generalized contraction was introduced by Belluce and Kirk (see [14]), who showed that such mappings form a subclass of mappings having diminishing orbital diameters. In [14] Kirk showed that generalized contractions occur naturally among mappings which are continuously Fréchet differentiable.

A Banach space X is called a Π_1 space provided that there exists a complete projection scheme for mappings from X to X, $\Gamma = (\{X_n\}, \{P_n\})$, such that $\|P_n\| \leq 1$

and $X_n \subset X_{n+1}$ for all n. This concept was originated by Lindenstrauss [18]. It turns out that Π_1 spaces are particularly suitable for studying the relationship between various types of Lipschitzian mappings and A-properness. For the rest of this section when we consider the A-properness of mappings of $A \subset X$ into X we shall assume our projection scheme Γ is Π_1.

When $S: X \to X$ is contractive with contractive constant $\alpha < 1$, Petryshyn [24] has shown that S is P_1-compact and in particular $I - S$ is A-proper. As yet it is unknown whether a mapping of the form $T = I - S: \bar{B}(0, 1) \subset X \to X$ is A-proper when S is contractive and X is a general Π_1 Banach space. However, if one imposes some additional hypotheses then one may guarantee A-properness.

Nussbaum has shown [19] that if the contraction constant α is such that $\alpha < \frac{1}{2}$ or if S may be extended to a contraction on $\bar{B}(0,2)$, then $I - S$ is A-proper on $\bar{B}(0, 1)$. Clearly if there exists a nonexpansive retraction onto the ball $\bar{B}(0, 1)$, then S may be extended to a contraction on $\bar{B}(0,2)$, and in fact to all of X. Unfortunately, there do not always exist nonexpansive retractions onto the unit ball [10]. Nussbaum also showed that if X had a certain geometric property, the ball intersection property (see [21]), then $I - S: \bar{B} \subset X \to X$ is A-proper when $S: \bar{B}(0, 1) \subset X \to X$ is contractive, and from this it was concluded that if $X = l^p$, $1 < p < \infty$, then $I - S: \bar{B}(0, 1) \subset X \to X$ is A-proper.

Petryshyn [27] has shown that when X is reflexive and has a single valued weakly continuous duality mapping then $I - S: \bar{B}(0, 1) \subset X \to X$ is A-proper, and consequently it also follows from his results that our question is answered affirmatively if $X = l^p$ with $1 < p < \infty$. It was also shown by Petryshyn [27] that if one assumes that S, in addition to being contractive, is weakly continuous, then $I - S: \bar{B}(0, 1) \subset X \to X$ is A-proper.

We shall now prove an A-properness result for generalized contractions defined on a closed convex subset of a reflexive Π_1 Banach space. Using an argument essentially modelled after Kirk [15], Wong [36] extended the result of Petryshyn [24] which stated that a mapping of the form $I - S: X \to X$, where S is a contraction, is A-proper. Wong proved this result for generalized contractions defined on all of X. We will now prove this result for generalized contractions defined only on a closed bounded convex subset of X. However, we are not able to show A-properness for all points, but only at specific points.

To clarify this remark let us introduce the following definition.

Definition 4.2. *Let X and Y be Banach spaces with $\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$ a complete projection scheme for mappings from X to Y. Suppose $T: D \subset X \to Y$ and $y_0 \in Y$. Then T is said to be A-proper at y_0 provided the following condition holds: if $\langle n_k \rangle$ is a sequence of integers with $\langle x_{n_k} \rangle \subset X$ a corresponding bounded sequence such that $x_{n_k} \in D \cap X_{n_k}$ for each k, and if $Q_{n_k} T(x_{n_k}) \to y_0$, then $\langle x_{n_k} \rangle$ has a subsequence which converges to $x \in D$ and $T(x) = y_0$.*

As we shall see, the above restricted definition of A-properness will be useful. Our technique of proof in the next theorem is modelled upon a proof used by Kirk to prove a theorem for mappings with diminishing orbital diameters.

Theorem 4.1. *Let X be a reflexive Π_1 Banach space. Let D be a closed convex subset of X, with S: $D \subset X \to X$ a generalized contraction. Let $y_0 \in X$ be such that $S(x) + y_0 \in D$ for all x in D. Then $T = I - S$: $D \subset X \to X$ is A-proper at y_0.*

Proof. Let $\langle k_n \rangle$ be a sequence of positive integers, with a corresponding sequence $\langle x_{k_n} \rangle \subset X$, such that $\langle x_{k_n} \rangle$ is bounded, $x_{k_n} \in D \cap X_{k_n}$ for each n, and $x_{k_n} - P_{k_n} S(x_{k_n}) \to y_0$. For the sake of notational convenience replace k_n by n. We shall now show $\langle x_n \rangle$ is a Cauchy sequence.

Let $R = \{r > 0 \mid$ there exists $n \in N$ with $D \cap \{\bigcap_{k \geq n} \bar{B}(x_k, r)\} \neq \varnothing\}$. Then $R \neq \varnothing$, since $\langle x_n \rangle$ is bounded. Consequently we may define $r_0 = \inf\{r \mid r \in R\}$.

We claim $r_0 = 0$, and our proof will be by contradiction. Hence, suppose $r_0 > 0$.

For each $\varepsilon > 0$, let

$$C_\varepsilon = D \cap \bigcup_{n \geq 0} \bigcap_{i=n}^{\infty} \bar{B}(x_i, r + \varepsilon).$$

Now note that the family $\{C_\varepsilon \mid \varepsilon > 0\}$ is a collection of nonempty weakly compact sets having the finite intersection property, and hence $\bigcap_{\varepsilon > 0} C_\varepsilon \neq \varnothing$. Choose $x \in \bigcap_{\varepsilon > 0} C_\varepsilon$. Now choose $\varepsilon > 0$ and $\delta > 0$ such that $\beta = \alpha(x)(r_0 + 2\varepsilon) + \delta < r_0$, and finally choose n_0 such that for $n \geq n_0$, $\|x_n - P_n(S(x_n) + y_0)\| < \delta/2$ and $\|S(x) + y_0 - P_n(S(x) + y_0)\| < \delta/2$. Consequently for $n \geq n_0$ we have

$$\begin{aligned} \|S(x) + y_0 - x_n\| &\leq \|S(x) + y_0 - P_n(S(x) + y_0)\| \\ &\quad + \|P_n(S(x) + y_0) - P_n(S(x_n) + y_0)\| \\ &\quad + \|P_n(S(x_n) + y_0) - x_n\| \\ &\leq \delta + \|S(x) + y_0 - (S(x_n) + y_0)\| \\ &\leq \delta + \alpha(x) \cdot \|x - x_n\| \\ &\leq \delta + \alpha(x) \cdot (r_0 + \varepsilon) = \beta < r_0. \end{aligned}$$

Hence it follows that $S(x) + y_0 \in D \cap \{\bigcap_{k \geq n_0} B(x_n, \beta)\}$. This contradicts the definition of r_0. Thus our assumption that $r_0 > 0$ is untenable and we conclude $r_0 = 0$. Hence it follows that for each $\varepsilon > 0$ we can find k such that $\|x_n - x_m\| \leq \varepsilon$, for all $n, m \geq k$, and consequently $\langle x_n \rangle$ is Cauchy.

Let $x_0 \in D$ be such that $x_n \to x_0$. Then since S is continuous, $x_0 - S(x_0) = y_0$. Q.E.D.

We shall now use Theorem 4.1 to prove a fixed point theorem and a surjectivity theorem for generalized contractions. Furthermore, these results will be of a constructive nature. The following concept, whose relationship to A-properness was studied by Petryshyn [25], will be needed.

Definition 4.3. *Let X and Y be Banach spaces with $\Gamma = (\{X_n\}, \{P_n\}, \{Y_n\}, \{Q_n\})$ a complete projection scheme for mappings from X to Y. Suppose $D \subset X$ and $T: D \to Y$. If $y \in Y$, consider the equations*

$$(4.1) \qquad T(x) = y \qquad (x \in D),$$

$$(4.2) \qquad Q_n T(x) = Q_n(y) \qquad (x \in D_n = D \cap X_n).$$

Then equation (4.1) *is said to be uniquely strongly projectionally solvable provided there exists $n_0 \in N$ such that for each $n \geq n_0$, equation* (4.2) *has a unique solution x_n, and the sequence $\langle x_n \rangle$ converges strongly to x_0, with x_0 the unique solution of $T(x) = y$ $(x \in D)$.*

Finally, the following result of Petryshyn and Tucker [31] will be necessary.

Proposition 4.1. *Let X be a Π_1 Banach space and suppose $D \subset X$ is open, bounded, and contains the origin. Suppose $T: \overline{D} \subset X \to X$ is continuous, bounded, P_1-compact, and satisfies*

$$\Pi_1^{<}: \quad \textit{if } T(x) = \lambda x, \textit{ with } x \in \dot{D}, \textit{ then } \lambda < 1.$$

Then there exists an $n_0 \in N$ such that for each $n \geq n_0$ there exists an $x_n \in D \cap X_n$ with $x_n - P_n T(x_n) = 0$.

Furthermore, $\langle x_n \rangle$ has a subsequence which converges to $x_0 \in D$ and $x_0 - T(x_0) = 0$.

Remark 4.1. An examination of the proof of the above result reveals that one may weaken the hypothesis that T is P_1-compact to the requirement that $\lambda I - T$ is A-proper at 0 for each $\lambda \geq 1$.

Theorem 4.2. *Let X be a reflexive Π_1 Banach space, and suppose $D \subset X$ is open, convex, and bounded. Assume $T: \overline{D} \to \overline{D}$ is a generalized contraction. Then T has a unique fixed point x_0 in $\overline{D}$ and if $x_0 \notin \dot{D}$ then the equation*

$$x - T(x) = 0, \qquad x \in D,$$

is uniquely strongly projectionally solvable.

Proof. Without loss of generality we may assume that $0 \in D$. If there exists an $x_0 \in \dot{D}$ such that $T(x_0) = x_0$, then since $I - T$ is one-to-one our theorem is proven.

Thus we assume T does not have a fixed point on $\dot{D}$. Then T satisfies the condition $\Pi_1^{<}$ of Proposition 4.2. Since $T: \overline{D} \to \overline{D}$ is a generalized contraction we see that for $0 < \beta \leq 1$ βT is a generalized contraction and $\beta T: \overline{D} \to \overline{D}$. Hence, by Theorem 4.1, $I - \beta T: \overline{D} \to X$ is A-proper at 0, and consequently $\lambda I - T$ is A-proper at 0 for each $\lambda \geq 1$. Thus, by invoking Proposition 4.1 and Remark 4.1, we can find $n_0 \in N$ such that for each $n \geq n_0$ there is an $x_n \in D \cap X_n$ such that $x_n - P_n T(x_n) = 0$. Since X is a Π_1 space, $I - T_n: \overline{D} \to X_n$ is one-to-one and hence the fixed point of T_n, for each $n \geq n_0$, is unique. Since $I - T$ is A-proper at 0, and $I - T$ is also one-to-one, there exists an $x_0 \in \overline{D}$ such that $x_n \to x_0$ with $x_0 - T(x_0) = 0$. By assumption T has no fixed points in $\dot{D}$, and thus $x_0 \in D$. Q.E.D.

Remark 4.2. When $D = B(0,r) \subset X$ where X is a Π_1 Banach space and $T\colon \overline{B} \to \overline{B}$, then $T_n\colon \overline{B}(0,r) \cap X_n \to \overline{B}(0,r) \cap X_n$ for each n, and hence it follows that in Theorem 4.2 the fixed point is obtained in a constructive fashion even if it lies on $\dot{D}$.

We may compare the above result to the Browder-Kirk-Göhde theorem. We have replaced the uniform convexity condition on X by the requirement that X be reflexive and Π_1, while strengthening the nonexpansiveness of T to T being a generalized contraction.

We note that while the strong projectional solvability aspect of the previous result is new, the existence part follows from a result of Kirk [15].

From Theorem 4.1 it follows immediately that if X is a reflexive Π_1 Banach space and $S\colon X \to X$ is a generalized contraction, then S is P_1-compact. Using this observation together with Proposition 4.1 we obtain the following constructive surjectivity theorem which generalizes the well-known result that if $T\colon X \to X$ is a contraction then $I - T$ is surjective.

Theorem 4.3. *Let X be a reflexive Π_1 Banach space, and suppose $T\colon X \to X$ is a generalized contraction. Then for each $y \in X$, the equation*

$$x - T(x) = y \qquad (x \in X)$$

is uniquely strongly projectionally solvable, with its solution lying in $\overline{B}(0, r_y)$, where

$$r_y = \|T(0) + y\|/(1 - \alpha(0)).$$

Proof. Let $y \in X$. Define $\tilde{T}\colon X \to X$ by $\tilde{T}(x) = T(x) + y$ for $x \in X$. Then $\tilde{T}$ is a generalized contraction and clearly to obtain our conclusion it suffices to show that

$$x - \tilde{T}(x) = 0, \qquad x \in X$$

is uniquely projectionally solvable, with solution lying in $\overline{B}(0, r_y)$. Let $r > r_y$. We shall show $\tilde{T}$ satisfies $\Pi_1^{<}$ on $\dot{B}(0,r)$. Let $x \in \dot{B}(0,r)$ and assume $\tilde{T}(x) = \lambda x$ with $\lambda \geq 1$. Then $\|\tilde{T}(x) - \tilde{T}(0)\| \leq \alpha(0)\|x\|$, and thus $\|\lambda x - \tilde{T}(0)\| \leq \alpha(0)\|x\|$. Hence, since $\lambda \geq 1$, $\lambda\|x\| - \|\tilde{T}(0)\| \leq \lambda \cdot \alpha(0)\|x\|$. Therefore

$$\lambda\|x\| \leq \|\tilde{T}(0)\|/(1 - \alpha(0)) = r_y.$$

Since $\|x\| > r_y$, we have a contradiction. Thus our assumption that $\lambda \geq 1$ is untenable. We conclude that $\tilde{T}$ satisfies $\Pi_1^{<}$ on $\dot{B}(0,r)$. We can now proceed, as in Theorem 4.2, to use Proposition 4.1 to show the unique projectional solvability of $x - \tilde{T}(x) = 0$, $x \in X$. Finally, since the solution lies in $B(0,r)$ for all $r > r_y$, we see that the solution lies in $\bar{B}(0,r_y)$. Q.E.D.

Kirk [15] has obtained a fixed point theorem for mappings which are formed by intertwining generalized contractions with uniformly strongly continuous mappings. We shall now obtain the main theorem of [15] in a constructive manner. To do so we will need the following proposition, a result which is implicitly contained in the proof of a fixed point theorem of Wong [36].

Proposition 4.2. *Let X be a reflexive Banach space and suppose $F \subset X$ is a weakly closed set. Suppose V: $F \times F \to Y$, where Y is a Banach space, satisfies:*

(1) $V(\cdot,x)$: $F \subset X \to Y$ is A-proper for each $x \in F$.

(2) If $\langle y_n \rangle \subset F$ is such that $y_n \rightharpoonup y$ and $\langle x_n \rangle \subset F$ is bounded, then $V(x_n,y_n) - V(x_n,y) \to 0$. Then the mapping T: $F \subset X \to Y$, defined by $T(x) = V(x,x)$ for each $x \in F$, is A-proper.

Proof. Let $\langle k_n \rangle$ be a sequence of integers with $\langle x_{k_n} \rangle \subset F$ a corresponding bounded sequence such that $x_{k_n} \in F \cap X_{k_n}$ for each n and $\langle Q_{k_n} T(x_{k_n}) \rangle \to g \in Y$. By reflexivity of X, we may assume that $x_{k_n} \rightharpoonup x_0$ and $x_0 \in F$, since F is weakly closed. However, by condition (2) we see that

$$Q_{k_n} V(x_{k_n}, x_{k_n}) - Q_{k_n} V(x_{k_n}, x_0) \to 0,$$

and hence $Q_{k_n} V(x_{k_n}, x_0) \to g$. Since $V(\cdot, x_0)$ is A-proper we may select a subsequence of $\langle x_{k_n} \rangle$ which converges strongly to some $z \in F$, with $V(z,x_0) = g$. But this subsequence also converges weakly to x_0 and hence $x_0 = z$, and $V(x_0,x_0) = T(x_0) = g$. Q.E.D.

Corollary 4.1. *Let X be a reflexive Π_1 Banach space, and let $F \subset X$ be closed and convex. Suppose V: $X \times F \to X$ satisfies the following conditions:*

(1) For each $y \in F$, $V(\cdot,y)$: $X \to X$ is a generalized contraction.

(2) If $\langle x_n \rangle \subset X$ is bounded and $\langle y_n \rangle \subset F$ is such that $y_n \rightharpoonup y_0$, then $V(x_n,y_n) - V(x_n,y_0) \to 0$. Then T: $F \subset X \to X$, defined by $T(x) = V(x,x)$ for $x \in F$, is P_1-compact.

Mappings of the form described in Corollary 4.1 have been termed strongly semicontractive by Kirk [15]. It is clear that when X is as above and $F \subset X$ is closed, convex, and bounded, that any strongly semicontractive mapping T: $F \to X$ is continuous and bounded. Consequently, we may combine Proposition 4.1 and Corollary 4.1 to obtain the following fixed point theorem.

Theorem 4.4. *Let X be a reflexive Π_1 Banach space, with $D \subset X$ bounded, open, with $0 \in D$. Then if T: $\bar{D} \subset X \to X$ is strongly semicontractive and satisfies $\Pi_1^<$ on $\dot{D}$, there exists some $x_0 \in D$ such that $x_0 - T(x_0) = 0$. Furthermore, one may find a sequence of integers $\langle k_n \rangle$ and $x_{k_n} \in D_{k_n}$ such that*

$$x_{k_n} - P_{k_n} T(x_{k_n}) = 0, \quad \textit{for each } n,$$

and $x_{k_n} \to x_0$.

In [4] Browder proved a result similar to the above theorem, where reflexivity was replaced by the stronger condition of uniform convexity, while the condition that $V(\cdot,y)\colon X \to X$ be a generalized contraction for each $y \in \overline{D}$ is replaced by the condition that $V(\cdot,y)\colon X \to X$ is nonexpansive for each $y \in \overline{D}$.

In [3] Browder also obtained some other fixed point theorems for intertwining mappings, under the assumption that the space considered was equipped with a weakly continuous single valued duality mapping.

Our next theorem introduces a new class of A-proper mappings.

Theorem 4.5. *Let X be a reflexive Π_1 Banach space and suppose $T\colon X \times X \to X$ satisfies the following conditions:*

(1) *$T(x,\cdot)$ is strongly continuous for each $x \in X$.*

(2) *For each ball $B = B(0,r) \subset X$ there exists an $\alpha = \alpha(B)$ with $0 < \alpha < 1$ and*

$$\|T(y,x) - T(z,x)\| \le \alpha \cdot \|y - z\|, \quad \textit{for all } x, z \in \overline{B} \textit{ and } y \in X.$$

Suppose $S\colon X \to X$ is defined by $S(x) = T(x,x)$, for all $x \in X$. Then S is P_1-compact.

Proof. It suffices to show that $I - S$ is A-proper. Let $\langle k_n \rangle$ be a sequence of integers and $\langle x_{k_n} \rangle$ a corresponding bounded sequence in X, with $x_{k_n} \in X_{k_n}$ for each n, and such that

$$\langle x_{k_n} - P_{k_n} S(x_{k_n}) \rangle \to g \in X.$$

As usual, for convenience of notation we replace k_n by n and also assume $x_n \rightharpoonup x$. Now the mapping $I - T(\cdot,x)\colon X \to X$ is onto and hence we may choose $y \in X$ such that $y - T(y,x) = g$. We claim that $x_n \to y$. Well,

$$\begin{aligned} x_n - y = x_n - P_n T(x_n,x_n) - y + P_n T(y,x) + P_n T(x_n,x_n) \\ - P_n T(y,x) + P_n T(y,x_n) - P_n T(y,x_n). \end{aligned}$$

Hence

$$\begin{aligned} \|x_n - y\| &\le \|x_n - P_n S(x_n) - y + P_n(y) - P_n(y) + P_n T(y,x)\| \\ &\quad + \|P_n(T(y,x_n) - T(y,x))\| + \|P_n(T(y,x_n) - T(x_n,x_n))\| \\ &\le \|x_n - P_n S(x_n) - g\| + \|P_n(y) - y\| \\ &\quad + \|T(y,x_n) - T(y,x)\| + \alpha \cdot \|y - x_n\| \end{aligned}$$

where $\alpha = \alpha(B(0,r))$, where $B(0,r) \supset \{x_n\}$.

Now by condition (1) we know that $\|T(y, x_n) - T(y, x)\| \to 0$. Clearly the first two terms converge to 0. Hence $(1 - \alpha) \cdot \|x_n - y\| \to 0$, or $x_n \to y$. Since $x_n \rightharpoonup x$, we see $x = y$ and $x - S(x) = g$. Q.E.D.

Proposition 4.1 may be combined with Theorem 4.5 to produce the following constructive version of a fixed point theorem of Webb [35].

Theorem 4.6. *Let X be a reflexive Π_1 Banach space, with $D \subset X$ open, bounded, convex, and containing the origin. Let $S: X \to X$ be such that $S(x) = T(x, x)$, where T satsifies the conditions of Theorem 4.5. Suppose, furthermore, that S satisfies $\Pi_1^<$ on $\dot{D}$. Then there exists $x_0 \in D$ such that $x_0 - S(x_0) = 0$, and a subsequence $x_{k_n} \in D_{k_n}$ with $x_{k_n} - P_{k_n} S(x_{k_n}) = 0$ for each n and $x_{k_n} \to x_0$.*

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